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COMPLETELY CONDITIONALLY PERMUTABLE SUBGROUP AND *p*-SUPERSOLUBILITY OF FINITE GROUPS

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Abstract

In this paper, we research p-supersolubility of finite groups. We determine the structure of some groups by using the completely conditionally permutable subgroups. We obtain some sufficient or necessary and sufficient conditions of a finite group is p-supersolvable.

1. Introduction

All groups considered in this paper are finite.

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Let H and T be subgroups of a group G. It is well known that H is called *permutable with* T, if $HT = TH$, H is said to be *permutable in* G, if H is permutable with all subgroups of G, and H be *s-permutable in* G, if $HP = PH$ for all Sylow subgroups P of G.

The permutable subgroups have many interesting properties. For example, Ore [9] proved that every permutable subgroups H of a group G is subnormal in G. However, for two subgroups H and T of a group G, may be they are not permutable but there exists an element $x \in G$ such that $HT^x = T^x H$. Guo et al. [4, 5] introduce the concepts of conditionally permutable subgroups and completely conditionally permutable subgroups. With these concepts, some new elegant results have been obtained [6-8, 11]. In this paper, we determine the structures of some groups by using the completely conditionally permutable subgroups. Some new criterions of p-supersolubility of some finite groups will be given and some known results are generalized.

We use "*cc*-permutable" to denote "completely conditionally permutable". As usual, we denote a maximal subgroup M of G by $M <\cdot G$ and a minimal normal subgroup A of G by $A \cdot \triangleleft G$. All unexplained notions and terminologies are standard, see [3] and [13].

2. Preliminaries

We cite here some known results which are useful in the later.

Definition 1 ([5]). Let G be a group. Suppose $H \leq G$ and $T \leq G$. Then

(1) H is called *cc-permutable with* T *in* G, if there exists some $x \in \langle H, T \rangle$ such that $HT^x = T^x H$, where $\langle H, T \rangle$ is the subgroup of G generated by H and T.

(2) H is called *cc-permutable in* G, if for every subgroup K of G, there exists some $x \in \langle H, K \rangle$ such that $HK^x = K^x H$.

Lemma 1 ([3]; Theorem 1.9.4). *The following conditions are equivalent*:

(1) *G is p-supersolvable*;

(2) *G is p-solvable and the index of every maximal subgroup of G either equal to p or be p'- number.*

Lemma 2 ([3]; Theorem 1.7.7)**.** *Let G be π'- solvable group. Then there at least exists one π'- Hall subgroup $G_{\pi'}$ of G, and for every π'- subgroup A of G, there exists some $x \in G$ such that $A^x \subseteq G_{\pi'}$. In particular, any two π'- Hall subgroups of G are conjugate in G.*

Lemma 3 ([3]; Theorem 1.7.6)**.** *Let G be π- solvable group. Then there at least exists one π- Hall subgroup G_{π} of G, and for every π- subgroup A of G, there exists some $x \in G$ such that $A^x \subseteq G_{\pi}$. In particular, any two π- Hall subgroups of G are conjugate in G.*

Lemma 4 ([4])**.** *Let G be a group. Suppose that $N \lhd G$ and $H \leq G$. Then*

(1) *If $N \leq T \leq G$ and H is cc-permutable with T in G, then HN / N is cc-permutable with T / N in G / N*;

(2) *Assume that $N \leq H$ and $T \leq G$, if H / N is cc-permutable with TN / N in G / N, then H is cc-permutable with T in G*;

(3) *Assume that $T \leq M \leq G$ and $H \leq M$, if H is cc-permutable in G, then H is cc-permutable with T in M.*

Lemma 5 ([2]; Theorem 1.8)**.** *Let G be p-solvable and outer p-supersolvable group. Then $G = AN$ and $A \cap N = 1$, where $A < \cdot G$, $N \cdot \lhd G$ and $|N| = p^{\alpha}$, $\alpha > 1$.*

Lemma 6 ([10]; Lemma 2)**.** *Let G be a group, if there exist subgroups M and K of G such that $G = MK$, then $G = M^x K^y$ for any $x, y \in G$.*

Lemma 7 ([1]; Theorem 2)**.** *If $G = AB$ is the product of two supersoluble subgroups A and B of G such that A permutes with every maximal subgroup of B, and B permutes with every maximal subgroup of A, then G is solvable group.*

3. Main Result

Theorem 1. *Let G be a p-solvable group. Then the following conditions are equivalent*:

(i) *G is p-supersolvable*;

(ii) *Every maximal subgroup of G with the index of p^{α} is cc-permutable in G, where α is an integer*;

(iii) *Every maximal subgroup of G with the index of p^{α} is cc-permutable with every maximal subgroup of Sylow p-subgroup of G in G*;

(iv) *Every maximal subgroup of G is cc-permutable with every maximal subgroup of Sylow p-subgroup of G in G.*

Proof. (i) $\Rightarrow$ (ii)

Let G be p-supersolvable group and M is a maximal subgroup of G, where $|G : M| = p^{\beta}$. It is clear that $|G : M| = p$ by Lemma 1. Obviously, we know that $\langle M, K\rangle = M$ or $\langle M, K\rangle = G$ for any subgroup K of G. If $\langle M, K\rangle = M$, then $K^x \subseteq M$ for any $x \in \langle M, K\rangle$ and $MK^x = M = K^x M$. If $\langle M, K\rangle = G$, let $K = K_p K_{p'}$ and $M = M_p M_{p'} = M_p G_{p'}$, $K_p \in Sylp(K)$, $M_p \in Sylp(M)$, $K_{p'} \in Hall_{p'}(K)$, $M_{p'} \in Hall_{p'}(M)$ and $G_{p'} \in Hall_{p'}(G)$. By Lemma 2, there exists some $x \in \langle M, K\rangle = G$ such that $K_{p'}^x \subseteq G_{p'} \subseteq M$. If $K_p^x \subseteq M$, then $MK^x = M = K^x M$. If $K_p^x \not\subseteq M$, then

$$G = K_p^x M = K^x M = MK^x.$$

All imply that M is cc-permutable in G.

(ii) $\Rightarrow$ (iii)

It is concluded from the definition of cc-permutable subgroups.

(iii) $\Rightarrow$ (iv)

Let G be a p-solvable group and every maximal subgroup of G with the index of p^{α} is cc-permutable with every maximal subgroup of Sylow p-subgroup of G in G.

For any maximal subgroup M of G, then $|G : M| = p^{\beta}$ or $|G : M|$ is a p'-number, where β is an integer. Set $P \in Sylp(G)$ and $P_1 < \cdot P$. If $|G : M| = p^{\beta}$, then M is cc-permutable with P_1 in G by the hypothesis. If $|G : M|$ is a p'-number, then $\langle M, P_1 \rangle = M$ or $\langle M, P_1 \rangle = G$ since $M < \cdot G$. Assume $\langle M, P_1 \rangle = M$. Clearly, for any $x \in \langle M, P_1 \rangle$, $P_1^x \subseteq M$ and $MP_1^x = M = P_1^x M$. Now suppose $\langle M, P_1 \rangle = G$. It is easy to see that $M = M_p M_{p'} = G_p M_{p'}$, where $M_p \in Sylp(M)$, $G_p \in Sylp(G)$ and $M_{p'} \in Hall_{p'}(M)$. By Lemma 3, there exists some $x \in \langle M, P_1 \rangle = G$ such that $P_1^x \subseteq G_p \subseteq M$. Hence, $MP_1^x = M = P_1^x M$. All imply that M is cc-permutable with P_1 in G.

(iv) $\Rightarrow$ (i)

Let G be a p-solvable group and every maximal subgroup of G is cc-permutable with every maximal subgroup of Sylow p-subgroup of G in G.

Assume that the proposition (i) is false and let G be a counterexample of a minimal order. Let $H \cdot \triangleleft G$, $M / H < \cdot G / H$, $P / H \in Sylp(G / H)$, and $P_1 / H < \cdot P / H$. If $P_0 \in Sylp(P)$ and $P_2 \in Sylp(P_1)$, then $M < \cdot G$, $P_0 \in Sylp(G)$ and $P_2 < \cdot P_0$. Hence, by the hypothesis, M is cc-permutable with P_2 in G. Clearly, $P_2 H / H = P_1 / H$ and $P_0 H / H = P / H$. By Lemma 4, P_1 / H is cc-permutable with M / H in G / H. This shows that the hypothesis holds on G / H.

Since G is p-solvable and outer p-supersolvable group, by Lemma 5, $G = AN$ and $A \cap N = 1$, where $A < \cdot G$, $N \cdot \triangleleft G$ and $|N| = p^{\alpha}$, $\alpha > 1$.

Let $N \in Sylp(G)$ and $N_1 < \cdot N$. By the hypothesis, A is cc-permutable with N_1 in G. Hence, there exists some $x \in \langle A, N_1 \rangle$ such that $D = N_1 A^x = A^x N_1$. If $D = G$, then $|G : A^x| = |N_1| = |G : A| = |N|$, this is a contradiction since $N_1 < \cdot N$. So, $D \neq G$, and $N_1 A^x = A^x$, since $A^x < \cdot G$. Then $N_1^{x^{-1}} \subseteq A \cap N = 1$, and $|N_1| = 1, |N| = p$, this is a contradiction. This induces that N is not a Sylow p-subgroup of G.

Let $A_p \in Sylp(A)$, by Lemma 3, there exists some subgroup $P \in Sylp(G)$ such that $A_p \subseteq P$. And there exists some subgroup P_1 of P such that $P_1 < \cdot P$ and $A_p \subseteq P_1$. By the hypothesis, A is cc-permutable with P_1 in G. So, there exists some $y \in \langle A, P_1 \rangle$ such that $B = P_1 A^y = A^y P_1$. Since $G = AN$, then there exists some $a \in A$ and $n \in N \subseteq P$ such that $y = an$. Hence, $B = P_1 A^n$ and $A_p^n \subseteq P_1^n = P_1$, since $P_1 \lhd P$. If $B = G$, then

$$P = P \cap P_1 A^n = P_1 (P \cap A^n) = P_1 A_p^n = P_1,$$

this is a contradiction. This implies that $B \neq G$. Thus $A^n < \cdot G$ and $B = A^n$, $P_1 \leq A^n$. So, $|G : A^n| = |G : A| = p = |N|$. This contradiction completes the proof.

Theorem 2. *Let G be a p-solvable group*, $G = AB$ *and* $A \in Sylp(G)$, $B \in Hall_{p'}(G)$. *If B is cc-permutable in G, then G is p-supersolvable.*

Proof. Assume that the assertion is false and G be a counterexample of a minimal order. Let $H \cdot \lhd G$. Then G / H is p-solvable group and $G / H = AH / H \cdot BH / H$, where $AH / H \in Sylp(G / H)$ and $BH / H \in Hall_{p'}(G / H)$. By the hypothesis and Lemma 4, BH / H is cc-permutable in G / H. This shows that the hypothesis holds on G / H.

Since G is p-solvable and outer p-supersolvable group, $G = MN$ and $M \cap N = 1$ by Lemma 5, where $M < \cdot G$, $N \cdot \lhd G$ and $|N| = p^{\alpha}$, $\alpha > 1$. Hence, $N \leq A$ and $A = A \cap G = A \cap NM = N(A \cap M)$. If $A \cap M = A$, then $N \leq A \subseteq M$, this is a contradiction. So, $A \cap M \neq A$ and there exists some subgroup T of G such that $T < \cdot A$ and $A \cap M \subseteq T$. By the hypothesis, B is cc-permutable in G. So, there exists some $x \in \langle B, T \rangle$ such that $BT^x = T^x B$. Hence,

$$G = AB = N(A \cap M)B = (NT)B = (NT)^x B = NBT^x.$$

This implies that either $BT^x = G$ or BT^x is a supplement of N in G. If $BT^x = G$, then $G = BT^x = BT$ by Lemma 6, and $A = A \cap BT = T$ $(A \cap B) = T$. If $BT^x \cap N = 1$, then $T^x \cap N = 1$ and $|N| = |A : T| = p$, since $A = NT$. This contradiction completes the proof.

Theorem 3. *Let G be a p-solvable group.* $G = AB$, *where A and B are p-supersolvable groups and* $(|A|, |B|) = 1$. *If A is cc-permutable with every maximal subgroup of B in G, and B is cc-permutable with every maximal subgroup of A in G, then G is a p-supersolvable group.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order.

Let $H \cdot \triangleleft G$. Obviously, G / H is a p-solvable group and $G / H = AH / H \cdot BH / H$, where AH / H and BH / H are p-supersolvable groups. Since $(|A|, |B|) = 1$,

$$(|AH / H|, |BH / H|) = (|A| / |A \cap H|, |B| / |B \cap H|) = 1.$$

Let $T / H < \cdot AH / H$. Then there exists a subgroup A_0 of G such that $A_0 < \cdot A$ and $A_0 H / H = T / H$. By the hypothesis, B is cc-permutable with A_0 in G. By Lemma 4, BH / H is cc-permutable with $A_0 H / H = T / H$ in G / H. Similarly, it can be proved that AH / H is cc-permutable with every maximal subgroup of BH / H in G / H. Thus, G / H satisfies the hypothesis and G / H is p-supersolvable.

Since G is a p-solvable and outer p-supersolvable group. By Lemma 5, $G = MN$, $M \cap N = 1$, and $|N| = p^{\alpha}$, $\alpha > 1$, where $N \cdot \triangleleft G$ and $M < \cdot G$. Since $(|A|, |B|) = 1$, without loss of generality, we may assume that $N \subseteq A$ and $B \subseteq M$. Then $A = A \cap G = A \cap NM = N(A \cap M)$. If

$A \cap M = A$, then $N \leq A \subseteq M$, this is a contradiction. Hence, $A \cap M \neq A$ and there exists a subgroup T of G such that $T < \cdot A$ and $A \cap M \subseteq T$. By the hypothesis, B is cc-permutable with T in G and there exists some $x \in \langle B, T \rangle$ such that $BT^x = T^x B$. Hence, $G = AB = N(A \cap M)B = (NT)^x B = NBT^x$. Then $BT^x \cap N = 1$ since $N \cdot \lhd G$ and N is a abelian group. So, $T^x \cap N = 1$ and $T \cap N = 1$. Then $|N| = |A : T| = p$, since $A = NT$, this is a contradiction. This implies that G is p-supersolvable group.

Corollary 4. *Let G be a p-solvable group. $G = AB$, where A and B are p-nilpotent groups and $(|A|, |B|) = 1$. If A is cc-permutable with every maximal subgroup of B in G and B is cc-permutable with every maximal subgroup of A in G, then G is p-supersolvable group.*

Corollary 5 ([12]; Theorem 3.1)**.** *A group G is supersoluble, if and only if $G = AB$ is the product of two supersoluble subgroups A and B of coprime orders, such that A permutes with every maximal subgroup of B, and B permutes with every maximal subgroup of A.*

Corollary 6 ([12]; Corollary 3.3)**.** *A group G is supersoluble, if and only if $G = AB$ is the product of two supersoluble subgroups A and B of coprime orders, such that every Sylow subgroup of B is permutable with every maximal subgroup of A, and every Sylow subgroup of A is permutable with every maximal subgroup of B.*

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